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Sequences with Finitely Many Negative Squares

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This work shows how the more general theory of sequences with finitely many negative squares can be reduced to the classical theory of moments via linear recursion. Specifically, if $f(n)$ is a real sequence with κ negative squares, then the main theorem asserts the existence of a minimal definitizing polynomial, i.e., a non-negative polynomial $q(x)=\sum_{i=0}^{2\kappa} a_i x^i$ of degree 2κ so that $g(n):=\sum_{i=0}^{2\kappa} a_i f(n+i)$ is a moment sequence, i.e., of the form $g(n)=\int x^n dv$ with $v\geqslant 0$. The question of uniqueness of a minimal definitizing polynomial is discussed. A Lévy–Khinchin-type integral formula for f is then explicitly obtained by solving a difference equation of the form $\sum_{i=0}^{2\kappa} a_i f(n+i)=\int x^n dv$. As a non-trivial application of the main theorem it is shown that many sequences of number theoretic importance, such as the sequence of primes, have infinitely many positive and infinitely many negative squares. Some of our results are contained in the work of Krein and Langer, but our proofs are different, and they are to a great extent independent of the theory of $\prod_\kappa$-spaces. © 1988 Academic Press, Inc.

INTRODUCTION

Hamburger's theorem asserts that the moment sequences are precisely the real sequences with 0 negative squares. These sequences, usually called positive definite, have a rich literature; cf. [1, 20]. Sequences with $\kappa>0$ negative squares have been studied in [15, 17]. An extensive theory of these sequences has been developed by Krein and Langer in [14] (announ-

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ced in [12]) in connection with a comprehensive study of extensions of hermitian operators in Π_κ-spaces; see [13].

Our investigations are completely independent of this work, and we use but a modest amount of Π_κ-spaces, and that only in a non-essential way. Our point of view is to use the main result (Theorem 3.1), stated in the abstract, to derive results about sequences with κ negative squares from the well-established theory of moments. The idea of finding a definitizing polynomial is not new, and it was apparently applied for the first time by Iohvidov and Krein in their papers [9], although in the context of Toeplitz sequences. The more general idea of finding a non-negative polynomial p, which is definitizing for a self-adjoint operator A in a space Π_κ, i.e., such that $[p(A)x, x] \geqslant 0$ for all x in the domain of the operator $p(A)$, goes back to Krein and Langer in [11]. Here $[\cdot, \cdot]$ is the indefinite scalar product on Π_κ, which by definition contains a maximal *negative* subspace of dimension κ. This means that (following [11]) we have changed the sign of the indefinite scalar product relative to the treatment in [10]. Reference [16] contains a characterization of the definitizing polynomials for A. The proof of the existence of a definitizing polynomial for A uses the fundamental theorem of Pontrjagin about the existence of a non-positive κ-dimensional invariant subspace for A.

The existence of a definitizing polynomial for a sequence with κ negative squares is established here by the separation theorem for convex sets in a finite-dimensional vector space. A converse of Theorem 3.1 is articulated in Corollary 1.4.

Uniqueness criteria for definitizing polynomials of minimal degree are also considered in Section 3. Sequences f which are *polynomially determinate* in the sense that $g(n) := \sum_{i=0}^{2d} a_i f(n+i)$ is a determinate moment sequence whenever $\sum_{i=0}^{2d} a_i x^i$ is a definitizing polynomial admit a unique definitizing polynomial of minimal degree which furthermore divides any definitizing polynomial. Exponentially bounded sequences and more generally quasi-analytic sequences are polynomially determinate; cf. Proposition 3.5 and Theorem 3.6. On the other hand, Theorem 3.7 exhibits a plethora of examples showing that uniqueness of definitizing polynomials of minimal degree does not hold in general.

Using Theorem 3.1 we deduce an integral representation for sequences with κ negative squares; cf. Theorem 3.9. Such a representation was established in [14, Sect. 2] based on integral representations for functions of class N_κ; see [13, Part I]. See also [17].

In Section 1 we start by giving an almost self-contained introduction to sequences with finitely many negative squares, and in Section 2 we collect some basic facts on what we call sequences of finite rank. These are precisely those sequences with finitely many non-zero squares or equivalently those which satisfy a linear recursion with constant coefficients

(Proposition 2.2). Moreover we show explicitly how to calculate the number of positive and negative squares for a sequence of finite rank. We conclude in Section 4 with the number theoretic application cited in the abstract above. Specifically, if f is an integer-valued sequence which is not of finite rank satisfying $|f(n)| \leqslant A \cdot B^n$ with $0 < B < 2$ then Theorem 4.1 asserts that f has infinitely many positive and infinitely many negative squares. The argument, based on generating functions, appeals to the main theorem and a theorem of Polyà. Proposition 4.2 shows that every sequence, such as the sequence of primes, which is asymptotic to $p(n)\log n$ for some polynomial p satisfies the hypothesis of Theorem 4.1.

1. Sequences with Finitely Many Negative Squares

We are concerned with the vector space $\Phi^{\mathbb{N}_0}$ ($\Phi = \mathbb{R}$ or $\mathbb{C}$) of real or complex sequences $f\colon \mathbb{N}_0 \to \Phi$, where $\mathbb{N}_0 = \{0, 1, 2, ...\}$. The set of polynomials in one variable with coefficients from Φ is denoted $\Phi[X]$. The unit shift operator $E\colon \Phi^{\mathbb{N}_0} \to \Phi^{\mathbb{N}_0}$ is defined by

$$Ef(n) = f(n+1), \qquad n \in \mathbb{N}_0, f \in \Phi^{\mathbb{N}_0}.$$

Then $p(E)f$ is a well-defined linear operator in $\Phi^{\mathbb{N}_0}$ for any polynomial $p \in \Phi[X]$. If $p(x) = \sum_{k=0}^{m} a_k x^k$ then

$$p(E)f(n) = \sum_{k=0}^{m} a_k f(n+k), \qquad n \in \mathbb{N}_0, f \in \Phi^{\mathbb{N}_0}.$$

In this way $\Phi^{\mathbb{N}_0}$ is organized as a $\Phi[X]$-module.

With a complex sequence f we associate the *Hankel matrix* of order m, $m = 0, 1, 2, ...$, defined by

$$H_m = H_m(f) = (f(i+j))_{0 \leqslant i, j \leqslant m}.$$

The Hankel matrices are symmetric, but hermitian only if f is real.

It is well known that a hermitian $(m+1) \times (m+1)$ matrix A has only real eigenvalues, and if $\kappa_+ = \kappa_+(A)$ and $\kappa_- = \kappa_-(A)$ denote respectively the number of positive and negative eigenvalues of A (counted with multiplicity) then $\kappa_+ + \kappa_-$ is equal to the rank of A, denoted $\mathrm{rk}(A)$. Furthermore κ_+ (resp. κ_-) is equal to the maximal dimension of positive (resp. negative) subspaces for A, considered as an operator on $\mathbb{C}^{m+1}$.

For a real sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ we put

$$\kappa_\pm = \kappa_\pm(f) = \sup_{m \geqslant 0} \kappa_\pm(H_m(f)),$$

and $\kappa_+(f)$ (resp. $\kappa_-(f)$) belonging to $\mathbb{N}_0 \cup \{\infty\}$ is called the *number of positive* (resp. *negative*) *squares* of f.

Clearly $\kappa_+(-f)=\kappa_-(f)$, and if f has $\kappa<\infty$ negative squares, then there exists $m_0\geqslant 0$ such that $H_m(f)$ has κ negative eigenvalues (counted with multiplicity) for $m\geqslant m_0$, and $H_m(f)$ has less than κ negative eigenvalues for $0\leqslant m<m_0$.

A sequence with 0 negative squares is also called *positive definite*, and by Hamburger's theorem these sequences are precisely the moment sequences for non-negative measures on $\mathbb{R}$. A sequence $f\colon \mathbb{N}_0\to\mathbb{R}$ is positive definite if and only if

$$|p|^2(E)f(0)=\sum_{i,j=0}^{m} f(i+j)\,c_i\,\overline{c_j}\geqslant 0$$

for any $p(x)=\sum_{i=0}^{m} c_i x^i\in\mathbb{C}[X]$.

The set of sequences with κ negative squares is a cone in $\mathbb{R}^{\mathbb{N}_0}$, but unless $\kappa=0$ it is not a convex cone.

In the following we will always assume $\kappa<\infty$ unless otherwise stated. As examples of sequences with at most one negative square we mention *conditionally positive definite sequences* defined as those sequences $f\colon\mathbb{N}_0\to\mathbb{R}$ for which

$$\sum_{i,j=0}^{m} f(i+j)\,c_i\overline{c_j}\geqslant 0 \tag{1.1}$$

for all $m\geqslant 1$ and all $c_0, c_1, ..., c_m\in\mathbb{C}$ such that $\sum_0^m c_i=0$. It is clearly enough to assume that (1.1) holds for all $c_0, ..., c_m\in\mathbb{R}$ with $\sum c_i=0$, and even enough to consider integers $c_0, ..., c_m$ with $\sum c_i=0$. For $p(x)=\sum_{i=0}^{m} c_i x^i$ we have $\sum c_i=0$ if and only if $p(x)=(x-1)\,q(x)$ for some $q\in\mathbb{C}[X]$ and therefore (1.1) is equivalent with

$$|\,p\,|^2(E)f(0)=|q|^2(E)(E-1)^2f(0)\geqslant 0$$

for all $q\in\mathbb{C}[X]$, i.e., with $(E-1)^2 f$ being positive definite.

The condition (1.1) means that the Hankel matrix H_m has the hyperplane $\{(c_0, c_1, ..., c_m)\in\mathbb{C}^{m+1}\,|\sum c_i=0\}$ as non-negative subspace, and therefore H_m has at most one negative eigenvalue, i.e., f has at most one negative square. Positive definite sequences are always conditionally positive definite. A sequence $f\colon\mathbb{N}_0\to\mathbb{R}$ is called *negative definite* iff $-f$ is conditionally positive definite. Negative definite sequences are extensively studied in [3], including an integral representation; see also Corollary 3.11 below.

Sequences with κ negative squares are linked to Π_κ-spaces, cf. [10], which are spaces with an indefinite inner product and a maximal positive subspace of dimension κ. We find it convenient to change the sign of the indefinite inner product, so there is a maximal *negative* subspace of dimension κ.

DEFINITION. Let $\kappa \in \mathbb{N}_0$. A *pre-Π_κ-space* is a complex vector space V together with a non-degenerate hermitian form $(\cdot, \cdot)$ from $V \times V$ to $\mathbb{C}$ such that:

(i) There is a κ-dimensional subspace $N \subseteq V$ which is negative, i.e., $(x, x) < 0$ for all $x \in N \setminus \{0\}$.

(ii) There is no negative subspace of V of dimension greater than κ.

From [10] it follows that the orthogonal complement $N^\perp$ of a negative subspace N of maximal dimension κ is a positive subspace, which in general might be of infinite dimension. If $N^\perp$ is complete with respect to the norm derived from the positive hermitian form on $N^\perp$, then V is called a Π_κ-space.

For any complex sequence f we define

$$A(f) = \{p(E) f \mid p \in \mathbb{C}[X]\},$$
$$(p(E) f, q(E) f)_f = (p\bar{q})(E) f(0).$$

Then $A(f)$ is a subspace of $\mathbb{C}^{\mathbb{N}_0}$ and $(\cdot, \cdot)_f$ is a non-degenerate sesquilinear form on $A(f)$. It is hermitian if and only if f is real.

PROPOSITION 1.1. *Let $f\colon \mathbb{N}_0 \to \mathbb{R}$ and $\kappa \in \mathbb{N}_0$. Then $\kappa_-(f) = \kappa$ if and only if $(A(f), (\cdot, \cdot)_f)$ is a pre-Π_κ-space.*

Proof. We have $f(n) = (f, E^n f)_f$, hence

$$\sum_{i,j=0}^{m} c_i \bar{c}_j f(i+j) = \left(\sum_{i=0}^{m} c_i E^i f, \sum_{i=0}^{m} c_i E^i f \right)_f,$$

and the result follows easily. ∎

An operator $T\colon V \to V$ in a pre-Π_κ-space is called *hermitian* if $(Tx, y) = (x, Ty)$ for all $x, y \in V$. The following simple result also links sequences with κ negative squares and pre-Π_κ-spaces.

PROPOSITION 1.2. *Let $(V, (\cdot, \cdot))$ be a pre-Π_κ-space and T a hermitian operator. If $\xi \in V$ is cyclic, i.e., $\{T^n \xi, n \in \mathbb{N}_0\}$ spans V, then*

$$f(n) = (\xi, T^n \xi), \qquad n \in \mathbb{N}_0 \tag{1.2}$$

has κ negative squares.

Conversely, if $f: \mathbb{N}_0 \to \mathbb{R}$ has κ negative squares, then there exist $(V, (\cdot, \cdot))$, T, and ξ as above such that (1.2) *holds.*

The proof is straightforward. If f has κ negative squares, we may define $V := A(f)$, $T := E$, and $\xi := f$. ∎

Remark. The representation $f(n)=(f, E^n f)_f$ with $V=A(f)=\{p(E)f \mid p\in\mathbb{C}[X]\}$ is called the *canonical representation.* This representation is unique up to isomorphism. In fact, if $(V_i, (\cdot,\cdot)_i, T_i, \xi_i)$, $i=1, 2$, are two quadruples such that

$$f(n)=(\xi_1, T_1^n \xi_1)_1=(\xi_2, T_2^n \xi_2)_2, \qquad n\in\mathbb{N}_0$$

and $T_i^n \xi_i$, $n\geqslant 0$, span V_i, then there is an isomorphism $\varphi\colon V_1\to V_2$ such that $(x, y)_1=(\varphi(x), \varphi(y))_2$ for $x, y\in V_1$ and $\varphi(\xi_1)=\xi_2$, $\varphi\circ T_1=T_2\circ\varphi$.

To see this notice that

$$\left(\sum c_i T_1^i \xi_1, \sum d_j T_1^j \xi_1\right)_1=\sum c_i \bar{d}_j f(i+j)$$

$$=\left(\sum c_i T_2^i \xi_2, \sum d_j T_2^j \xi_2\right)_2,$$

which shows that $\varphi\colon V_1\to V_2$ is well defined by

$$\varphi\left(\sum c_i T_1^i \xi_1\right)=\sum c_i T_2^i \xi_2$$

and possesses the desired properties.

In the following we will often use the fact that a non-negative polynomial $q\in\mathbb{R}[X]$ is of the form $q=|r|^2$ with $r\in\mathbb{C}[X]$.

THEOREM 1.3. *Let $f\colon\mathbb{N}_0\to\mathbb{R}$, let q be a non-negative polynomial of degree 2κ, and let $g:=q(E)f$. Then we have $\kappa_\pm(f)-\kappa\leqslant\kappa_\pm(g)\leqslant\kappa_\pm(f)$.*

Proof. We will prove the above inequalities for the case of minus sign. The other case follows by replacing f with $-f$. Let $r\in\mathbb{C}[X]$ be such that $|r|^2=q$. We define

$$A(f)=\{p(E)f \mid p\in\mathbb{C}[X]\}, \qquad W=\{(pr)(E)f \mid p\in\mathbb{C}[X]\},$$

and let $\varphi\colon A(f)\to A(f)/W$ denote the quotient mapping. Then dim $A(f)/W\leqslant\kappa$ because $\varphi(E^k f)$, $k=0, 1, ..., \kappa-1$, span $A(f)/W$. In fact, any polynomial $a\in\mathbb{C}[X]$ may be written $a=pr+b$, where b is of degree less than κ.

If $\psi\colon W\to A(g)$ denotes the linear map

$$\psi((pr)(E)f)=(p\,|r|^2)(E)f=p(E)\,g,$$

we have

$$(p(E)\,g, p(E)\,g)_g=|p|^2(E)\,g(0)=((pr)(E)f, (pr)(E)f)_f, \tag{1.3}$$

i.e., ψ is "unitary."

If $N \subseteq A(f)$ is a negative subspace with respect to $(\cdot, \cdot)_f$ then $\psi(N \cap W) \subseteq A(g)$ is a negative subspace with respect to $(\cdot, \cdot)_g$ by (1.3), which also shows that ψ is one-to-one on $N \cap W$. Therefore we have

$$\dim(N \cap W) = \dim \psi(N \cap W) \leqslant \kappa_-(g).$$

Using that $\varphi \mid N \colon N \to A(f)/W$ is a linear map with $\ker(\varphi \mid N) = N \cap W$, we find

$$\dim N = \dim(N \cap W) + \dim \varphi(N) \leqslant \kappa_-(g) + \kappa,$$

showing by Proposition 1.1 that $\kappa_-(f) \leqslant \kappa_-(g) + \kappa$.

In order to establish $\kappa_-(g) \leqslant \kappa_-(f)$, it is sufficient to prove that $\dim M \leqslant \kappa_-(f)$ for any subspace $M \subseteq A(g)$ which is negative with respect to $(\cdot, \cdot)_g$. By (1.3) we get that $\psi^{-1}(M)$ is a non-positive subspace of $A(f)$, i.e., $(x, x)_f \leqslant 0$ for all $x \in \psi^{-1}(M)$. By [10, Lemma 1.2] this implies that $\dim \psi^{-1}(M) \leqslant \kappa_-(f)$ and hence $\dim M \leqslant \dim \psi^{-1}(M) \leqslant \kappa_-(f)$. ∎

COROLLARY 1.4. *Let $f \colon \mathbb{N}_0 \to \mathbb{R}$ and let q be a non-negative polynomial of degree 2κ.*

If $q(E)f$ is positive definite then $\kappa_-(f) \leqslant \kappa$.

If f has κ negative squares, Corollary 1.4 suggests seeking a non-negative polynomial q such that $q(E)f$ is positive definite. A monic[1] polynomial with this property will be called *definitizing* for f. It is not clear at all that such a polynomial exists, but should it exist, then Corollary 1.4 implies that its degree must be at least 2κ. Note that if q is a definitizing polynomial for f, then so is qq_1 for any non-negative monic polynomial q_1. This follows by Theorem 1.3. Our main result of Section 3 (Theorem 3.1) establishes the existence of a definitizing polynomial of degree exactly 2κ. We call such a polynomial a *minimal definitizing polynomial* for f.

EXAMPLE 1.5. The sequences f_j, $j = 1, 2$, as defined by

$$f_1(n) = \begin{cases} a, & n = 0, \\ 1/n, & n \geqslant 1, \end{cases} \qquad f_2(n) = \begin{cases} a, & n = 0, \\ (n-1)!, & n \geqslant 1, \end{cases}$$

have exactly one negative square, independent of $a \in \mathbb{R}$. In fact, since

$$E^2 f_1(n) = \frac{1}{n+2} = \int_0^1 t^{n+1}\, dt, \qquad E^2 f_2(n) = (n+1)! = \int_0^\infty t^{n+1} e^{-t}\, dt,$$

we see that $E^2 f_1$ and $E^2 f_2$ are positive definite, so $\kappa_-(f_j) \leqslant 1$ for $j = 1, 2$ by Corollary 1.4. That they do in fact have one negative square follows by

[1] A polynomial is called *monic* if its leading coefficient is one.

contradiction, for if f_j had 0 negative squares, then there exist non-negative finite Radon measures μ_j on $\mathbb{R}$ such that

$$f_j(n) = \int t^n \, d\mu_j(t),$$

hence

$$E^2 f_j(n) = \int t^n t^2 \, d\mu_j(t), \qquad j = 1, 2.$$

The moment sequences $1/(n+2)$ and $(n+1)!$ are determinate (e.g., by Carleman's condition, cf. [20]) and therefore

$$t^2 \, d\mu_1(t) = t \, 1_{]0,1]}(t) \, dt, \qquad t^2 \, d\mu_2(t) = te^{-t} \, 1_{]0,\infty[}(t) \, dt,$$

but these equations lead to the contradictory conclusion

$$\mu_1(]0, \ 1]) = \mu_2(]0, \ 1]) = \infty.$$

In both cases $q(x) = x^2$ is a minimal definitizing polynomial.

2. Sequences of Finite Rank

For a polynomial $p \in \mathbb{C}[X]$ of degree d, an equation of the form $p(E)f = g$ is called a *linear difference equation of order d with constant coefficients*. Here $g \in \mathbb{C}^{\mathbb{N}_0}$ is given and the unknown sequence f will be sought from $\mathbb{C}^{\mathbb{N}_0}$. For a treatment of the classical theory of such equations see, e.g., [5].

A sequence $f \in \mathbb{C}^{\mathbb{N}_0}$ is said to be of *finite rank*, if it is a solution to an equation $p(E)f = 0$ for some $p \in \mathbb{C}[X]$, $p \neq 0$.

If f is of finite rank, there exists a unique monic polynomial p of lowest degree such that $p(E)f = 0$, called the *minimal* polynomial for f. Its degree is called the *rank* of f, denoted $\mathrm{rk}(f)$. Notice that if f is real, then the minimal polynomial p is also real.

Given $z_0 \in \mathbb{C}$ and $k \in \mathbb{N}_0$, the solutions to the equation $(E - z_0)^k f = 0$ are given by

$$f(n) = q(n) \, z_0^n, \tag{2.1}$$

where $q \in \mathbb{C}[X]$ is an arbitrary polynomial of degree $\leqslant k - 1$. Any sequence f of finite rank admits a unique decomposition

$$f(n) = \sum_{j=1}^{m} q_j(n) \, z_j^n,$$

where $z_1, ..., z_m$ are the different roots of the minimal polynomial p for f. Here q_j is a polynomial of degree $k_j - 1$, k_j denoting the multiplicity of the root z_j in p.

Given $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and $k \in \mathbb{N}_0$, the real solutions to the equation $|E - z_0|^{2k} f = 0$ are given by

$$f(n) = \mathrm{Re}(q(n)\, z_0^n), \tag{2.2}$$

where $q \in \mathbb{C}[X]$ is an arbitrary polynomial of degree $\leqslant k-1$. Putting $z_0 = \rho e^{i\theta}$, $\rho > 0$, $\theta \in \mathbb{R}$, and $q = q_1 - iq_2$ with $q_1, q_2 \in \mathbb{R}[X]$, then we have

$$f(n) = (q_1(n) \cos n\theta + q_2(n) \sin n\theta)\, \rho^n. \tag{2.3}$$

A real sequence f of finite rank admits a unique decomposition as sum of sequences of the form (2.2). Each component corresponds to a real root or a pair of complex conjugate roots of the minimal polynomial for f.

The following result is now easy to prove:

LEMMA 2.1. *Suppose $f \in \mathbb{R}^{\mathbb{N}_0}$ is a solution to the difference equations $p_i(E) f = 0$, $i = 1, 2$, where $p_1, p_2 \in \mathbb{R}[X]$ are relatively prime, then $f = 0$.*

Sequences f of finite rank can be characterized by $A(f)$ being of finite dimension. More precisely we have the following easily established result:

PROPOSITION 2.2. *For a sequence f: $\mathbb{N}_0 \to \mathbb{C}$ the following are equivalent:*

(i) *f is of finite rank.*

(ii) $\dim A(f) < \infty$.

If (i)–(ii) *hold then* $\mathrm{rk}(f) = \dim A(f)$, *and if in addition f is real then* $\mathrm{rk}(f) = \kappa_+(f) + \kappa_-(f)$.

We shall now give a description of the rank and the numbers $\kappa_\pm(f)$ for a real sequence f of finite rank.

We will first consider *elementary real sequences*, i.e., sequences of the form (2.2).

PROPOSITION 2.3. (i) *If $f(n) = q(n)\, x_0^n$, where $q(x) = \sum_{j=0}^{k-1} a_j x^j \in \mathbb{R}[X]$ is of degree $k-1$ and $x_0 \in \mathbb{R}$, then* $\mathrm{rk}(f) = k$ *and*

$$(\kappa_+(f), \kappa_-(f)) = \begin{cases} \left(\dfrac{k}{2}, \dfrac{k}{2}\right), & \text{if } k \text{ is even}, \\[2ex] \left(\dfrac{k+1}{2}, \dfrac{k-1}{2}\right), & \text{if } k \text{ is odd and } a_{k-1} > 0, \\[2ex] \left(\dfrac{k-1}{2}, \dfrac{k+1}{2}\right), & \text{if } k \text{ is odd and } a_{k-1} < 0. \end{cases}$$

(ii) *If* $f(n) = \mathrm{Re}(q(n)\, z_0^n) = q_1(n)\, \rho^n \cos(n\theta) + q_2(n)\, \rho^n \sin(n\theta)$, *where* $z_0 = \rho e^{i\theta} \in \mathbb{C} \backslash \mathbb{R}$, *and* $q = q_1 - iq_2, q_1, q_2 \in \mathbb{R}[X]$, *and* $\deg q = k-1$, *then* $\mathrm{rk}(f) = 2k$ *and* $(\kappa_+(f), \kappa_-(f)) = (k, k)$.

Proof. (i) The minimal polynomial is $p(x) = (x - x_0)^k$, so $\mathrm{rk}(f) = k$.

If k is even, then $((E - x_0)^{k/2})^2 f = 0$, so f and $-f$ have at most $k/2$ negative squares by Corollary 1.4, i.e., $\kappa_+, \kappa_- \leqslant k/2$, but since their sum is k, they are both equal to $k/2$.

If k is odd, then $(E - x_0)\, p(E) f = ((E - x_0)^{(k+1)/2})^2 f = 0$, so $\kappa_+, \kappa_- \leqslant \frac{1}{2}(k+1)$, $\kappa_+ + \kappa_- = k$, i.e., $\{\kappa_+, \kappa_-\} = \{(k-1)/2,\ (k+1)/2\}$. Now $g = (E - x_0)^{k-1} f$ is a solution to the equation $(E - x_0)\, g = 0$, i.e.,

$$(E - x_0)^{k-1} f(n) = a x_0^n, \qquad \text{where} \quad a \in \mathbb{R} \backslash \{0\}$$

($a = 0$ is impossible because $\mathrm{rk}(f) = k$).

If $a > 0$ then $(E - x_0)^{k-1} f$ is positive definite, so by Corollary 1.4, $\kappa_- \leqslant (k-1)/2$ and hence $(\kappa_+, \kappa_-) = ((k+1)/2,\ (k-1)/2)$. If $a < 0$ then the above applies to $-f$.

We finally claim that

$$a = (k-1)!\, a_{k-1} x_0^{k-1} \tag{2.4}$$

so a and a_{k-1} have the same sign, k being odd.

To see (2.4) we calculate

$$(E - x_0) f(n) = (q(n+1) - q(n))\, x_0^{n+1} = \tilde{q}(n)\, x_0^n$$

and $\tilde{q} \in \mathbb{R}[X]$ is of degree $k-2$ with leading coefficient $(k-1)\, a_{k-1}\, x_0$. By repeated applications of this procedure we get (2.4).

(ii) The minimal polynomial is $p(x) = |x - z_0|^{2k}$, so $\mathrm{rk}(f) = 2k$. By Corollary 1.4 it follows that $\kappa_+, \kappa_- \leqslant k$, and since $\kappa_+ + \kappa_- = 2k$ we have $\kappa_+ = \kappa_- = k$. ∎

In the above proof we have obtained the following:

COROLLARY 2.4. *If* $f: \mathbb{N}_0 \to \mathbb{R}$ *is a sequence of the form* (2.2) *with* κ *negative squares, then* $|E - z_0|^{2\kappa} f$ *is positive definite.*

In particular $|x - z_0|^{2\kappa}$ *is a minimal definitizing polynomial for* f.

Remark. A minimal definitizing polynomial is unique in the above case, since an elementary sequence is exponentially bounded; cf. Section 3.

PROPOSITION 2.5. *Let* $f: \mathbb{N}_0 \to \mathbb{R}$ *be a sequence of finite rank and let*

$$f = f_1 + \cdots + f_m$$

be the unique decomposition of f as sum of elementary real sequences. Then

(i) $\operatorname{rk}(f) = \operatorname{rk}(f_1) + \cdots + \operatorname{rk}(f_m)$,

(ii) $\kappa_\pm(f) = \kappa_\pm(f_1) + \cdots + \kappa_\pm(f_m)$.

Proof. Let p be the minimal polynomial for f and let $z_1, ..., z_m$ be the roots of p with $\operatorname{Im} z_j \geqslant 0$ such that f_j is associated with $z_j, j = 1, ..., m$. Equation (i) is obvious. The polynomial

$$q(x) = \prod_{i=1}^{m} |x - z_i|^{2\kappa_-(f_i)}$$

is such that $q(E)f$ is positive definite. In fact

$$q(E) f_i = \left(\prod_{\substack{j=1 \\ j \neq i}}^{m} |E - z_j|^{2\kappa_-(f_j)} \right) |E - z_i|^{2\kappa_-(f_i)} f_i$$

is positive definite by Corollary 2.4 and Theorem 1.3.

It follows by Corollary 1.4 that $\kappa_-(f) \leqslant \sum_{i=1}^m \kappa_-(f_i)$, and similarly $\kappa_+(f) \leqslant \sum_{i=1}^m \kappa_+(f_i)$. By addition we get

$$\operatorname{rk}(f) = \kappa_-(f) + \kappa_+(f) \leqslant \sum_{i=1}^{m} \kappa_-(f_i) + \kappa_+(f_i) = \sum_{i=1}^{m} \operatorname{rk}(f_i),$$

and since the equality sign holds, we have (ii). ∎

Results about sequences $f: \mathbb{N}_0 \to \mathbb{C}$ of finite rank go back to Kronecker and Frobenius. We recall the following characterization which will be used later. For a proof see [6, Chap. XVI, Sect. 10] or [8, p. 79].

PROPOSITION 2.6. *For a sequence $f: \mathbb{N}_0 \to \mathbb{C}$ the following conditions are equivalent:*

(i) *f is of finite rank.*

(ii) *The Hankel matrices H_m, $m \geqslant 0$, are of bounded rank.*

(iii) *The exists m_0 such that* $\det(H_m) = 0$ *for $m \geqslant m_0$.*

(iv) *The power series $\sum_0^\infty f(n) z^n$ has positive radius of convergence and represents a rational function $F(z)$.*

If any of the conditions (i) *through* (iv) *holds then*

$$\operatorname{rk}(f) = \max_{m \geqslant 0} (\operatorname{rk}(H_m)) = \min\{n \in \mathbb{N}_0 \mid \det(H_m) = 0 \ \forall m \geqslant n\},$$

and this number is also the number of poles (counted with multiplicity) of the rational function $(1/z)\, F(1/z)$.

Remark. The sequences f_j from Example 1.5 are not of finite rank as is easily seen, e.g., by Proposition 2.6(iv). It follows that they have infinitely many positive squares.

3. Existence of a Minimal Definitizing Polynomial

As a converse to Corollary 1.4 we will prove the following:

THEOREM 3.1. *Let $f: \mathbb{N}_0 \to \mathbb{R}$ be a sequence with κ negative squares. Then there exists a minimal definitizing polynomial for f, i.e., a non-negative monic polynomial q of degree 2κ such that $q(E)f$ is positive definite.*

Proof. Let C be the convex cone of Hankel matrices $H_\kappa(p(E)f+g)$, where p and g range over all non-negative polynomials and all positive definite sequences, respectively.

We claim that each matrix in C has at most κ negative eigenvalues. Indeed $H_\kappa(p(E)f)$ has at most κ negative eigenvalues by Corollary 1.4, and it is clear that this cannot be increased by adding a positive semi-definite matrix $H_\kappa(g)$. The Hankel matrix $H_\kappa(h_0)$, where $h_0(n) = -1/(n+1)$, has $\kappa+1$ negative eigenvalues since

$$\sum_{j,k=0}^{\kappa} h_0(j+k)\, c_j c_k = -\int_0^1 \left(\sum_{j=0}^{\kappa} c_j t^j\right)^2 dt < 0$$

for all $(c_0, c_1, ..., c_\kappa) \in \mathbb{R}^{\kappa+1} \setminus \{0\}$. It follows that $H_\kappa(h_0)$ does not belong to the closure of C in the ordinary topology on the vector space M of $(\kappa+1)\times(\kappa+1)$ matrices, so in particular C is contained in a closed half-space in M. This implies the existence of a matrix $(c_{ij}) \in M \setminus \{0\}$ such that

$$\sum_{i,j=0}^{\kappa} c_{ij}(p(E)f(i+j)+g(i+j)) \geqslant 0$$

for all non-negative polynomials p and all positive definite sequences g. For $g(n)=t^n$, $t\in\mathbb{R}$, and $p=0$ we get

$$q(t) := \sum_{i,j=0}^{\kappa} c_{ij}\, t^{i+j} = \sum_{k=0}^{2\kappa} \left(\sum_{i+j=k} c_{ij}\right) t^k \geqslant 0.$$

For $g=0$ and $p\geqslant 0$ we get

$$0 \leqslant \sum_{i,j=0}^{\kappa} c_{ij} p(E) f(i+j) = q(E)(p(E)f)(0) = p(E)(q(E)f)(0),$$

showing that $q(E)f$ is positive definite. By Corollary 1.4, q must be of degree $\geq 2\kappa$, so q has indeed degree 2κ, and $(1/c_{\kappa\kappa})\, q$ is a minimal definitizing polynomial for f. ∎

Remark. Let Π_κ be a Π_κ-space which is a completion of the pre-Π_κ-space $(A(f), (\cdot, \cdot)_f)$. Then E is a hermitian operator in Π_κ such that $f(n) = (f, E^n f)_f$, and we may chose a self-adjoint extension $\tilde{E}$ of E because E is real; cf. [10]. By the theorem of Pontrjagin there exists a κ-dimensional non-positive subspace $V \subseteq \mathrm{dom}(\tilde{E})$ such that $\tilde{E}(V) \subseteq V$. If p is a monic polynomial of degree κ such that $p(\tilde{E})$ vanishes on V, then it is easy to see that $|p|^2$ is definitizing for $\tilde{E}$ and in particular definitizing for f. This shows how Theorem 3.1 may be deduced from the theory of operators in Π_κ-spaces. This method was used for unitary operators in [9] and for self-adjoint operators in [11, 16].

Let $f\colon \mathbb{N}_0 \to \mathbb{R}$ be a sequence with κ negative squares and let $f(n) = (f, E^n f)_f$ be the canonical representation in $A(f)$; cf. Section 1. We show below that the existence of a definitizing polynomial for f is equivalent with the existence of a $\mathbb{C}[X]$-submodule of $A(f)$ which is also a non-negative subspace. Theorem 3.1 therefore ensures the existence of such a submodule. More precisely we have:

PROPOSITION 3.2. *Let $f\colon \mathbb{N}_0 \to \mathbb{R}$ have κ negative squares. If q is a definitizing polynomial for f and $|r|^2 = q$, then*

$$W_r = \{(pr)(E) f \mid p \in \mathbb{C}[X]\}$$

is a non-negative $\mathbb{C}[X]$-submodule of $A(f)$. Conversely, any non-zero and non-negative $\mathbb{C}[X]$-submodule of $A(f)$ has the form W_r for some monic polynomial $r \in \mathbb{C}[X]$, and furthermore $q = |r|^2$ is definitizing for f.

If q is a minimal definitizing polynomial and $|r|^2 = q$ then co-dim $W_r = \kappa$. *If furthermore* $\mathrm{rk}(f) = \infty$, *then W_r is a positive $\mathbb{C}[X]$-submodule.*

Proof. If $q = |r|^2$ is a definitizing polynomial for f, then

$$((pr)(E) f, (pr)(E) f)_f = |p|^2(E)\, q(E) f(0) \geq 0 \tag{3.1}$$

because $q(E) f$ is positive definite. This shows that the $\mathbb{C}[X]$-submodule W_r is non-negative.

Conversely, if $W \subseteq A(f)$ is a non-zero $\mathbb{C}[X]$-submodule then $I = \{p \in \mathbb{C}[X] \mid p(E) f \in W\}$ is a non-zero ideal in $\mathbb{C}[X]$, hence of the form $\{pr \mid p \in \mathbb{C}[X]\}$, where r may be chosen monic, and then clearly $W = W_r$. If W is furthermore non-negative, (3.1) shows that $q(E) f$ is positive definite, i.e., $q = |r|^2$ is a definitizing polynomial for f.

In the following we assume that q is a minimal definitizing polynomial so deg $q = 2\kappa$. Choose $r \in \mathbb{C}[X]$ so that $q = |r|^2$.

Let $\varphi: A(f) \to A(f)/W_r$ be the canonical map. Then $\varphi(E^j f)$, $j=0, 1, ..., \kappa-1$, span $A(f)/W_r$, cf. the proof of Theorem 1.3, so $\dim A(f)/W_r = \text{co-dim } W_r \leqslant \kappa$.

The proof is divided in two parts:

(a) $\text{rk}(f) = \infty$. In this case $\varphi(E^j f)$, $j = 0, 1, ..., \kappa - 1$, are independent for otherwise there exists $(\lambda_0, ..., \lambda_{\kappa-1}) \neq (0, ..., 0)$ such that $\sum \lambda_j \varphi(E^j f) = 0$, i.e., $t(E) f \in W_r$ with $t(x) = \sum \lambda_j x^j$. But this requires t to be a multiple of r, which is clearly impossible, since r is of degree κ.

To see that W_r is a positive subspace we consider $(\text{pr})(E) f \in W_r$ such that $|p|^2 q(E) f(0) = 0$. Then $|p|^2 q(E) f \equiv 0$ since it is a moment sequence. The rank of f being infinite we conclude that $|p|^2 q = 0$ hence $p = 0$ and $(pr)(E) f = 0$.

(b) $\text{rk}(f) < \infty$. From the general theory of Π_κ-spaces we know that $\dim W_r \leqslant \kappa_+(f)$ (see [10, p. 12]), and hence $\text{co-dim } W_r \geqslant \text{rk}(f) - \kappa_+(f) = \kappa_-(f) = \kappa$, so finally $\text{co-dim } W_r = \kappa$. ∎

The following example concerning W_r shows that "non-negative" cannot be sharpened to "positive" in the case of finite rank.

EXAMPLE 3.3. Let $f(n) = n + 1$. Then f has rank 2 and $(E-1)^2 f = 0$ so $q(x) = (x-1)^2$ is the minimal polynomial as well as a minimal definitizing polynomial. Furthermore $\kappa_-(f) = \kappa_+(f) = 1$.

In this case W_r consist of the constant sequences. They are isotropic vectors, and it is an easy exercise of linear algebra to show that W_r is the only one-dimensional submodule of $A(f)$.

As shown by Theorem 3.7 below, a minimal definitizing polynomial for f is not necessarily unique, but we will give some sufficient conditions for unicity.

DEFINITION. A sequence $f: \mathbb{N}_0 \to \mathbb{R}$ with κ negative squares is called *polynomially determinate* if $q(E) f$ is a determinate moment sequence for any definitizing polynomial q for f.

Remark. A determinate positive definite sequence is not necessarily polynomially determinate.

Indeed, if μ is a Nevanlinna extremal measure then

$$f(n) = \int x^n (1 + x^2)^{-1} \, d\mu(x)$$

is a determinate positive definite sequence, but $(E^2 + 1) f$ is indeterminate, so f is not polynomially determinate.

Concerning the above concepts see [1] or [20], in particular [20, p. 64].

In analogy with the Carleman condition for moment sequences we will introduce quasi-analytic sequences.

Let $(M_n)_{n \geq 0}$ be a sequence of positive numbers satisfying

(i) $M_0 = 1$,

(ii) $M_n^2 \leqslant M_{n-1} M_{n+1}$, $n \geqslant 1$ (M_n is log-convex).

With (M_n) we associate the sequence space $\mathscr{F}\{M_n\}$ of sequences $f\colon \mathbb{N}_0 \to \mathbb{R}$ satisfying

$$|f(n)| \leqslant AB^n M_n, \qquad n \geqslant 0,$$

where $A, B > 0$ may depend on f.

In analogy with quasi-analytic classes of functions we say that $\mathscr{F}\{M_n\}$ is a *quasi-analytic sequence space*, if the following two equivalent conditions on (M_n) hold:

(iii) $\sum_{n=0}^{\infty} 1/\sqrt[n]{M_n} = \infty$,

(iv) $\sum_{n=1}^{\infty} M_{n-1}/M_n = \infty$.

(For a proof of (iii) $\Leftrightarrow$ (iv) see [18].)

We say that $f\colon \mathbb{N}_0 \to \mathbb{R}$ is *quasi-analytic* if it belongs to some quasi-analytic sequence space.

A quasi-analytic sequence corresponding to $M_n = 1$, $n \geqslant 0$, will be called *exponentially bounded.*

LEMMA 3.4. *If $f\colon \mathbb{N}_0 \to \mathbb{R}$ is quasi-analytic and $p \in \mathbb{R}[X]$ then $p(E)f$ is quasi-analytic.*

Proof. Assume $|f(n)| \leqslant AB^n M_n$ and $p(x) = \sum_{j=0}^{d} a_j x^j$, where $\{M_n\}$ satisfies (i)–(iv) above. Then

$$|p(E) f(n)| \leqslant AB^n \sum_{j=0}^{d} |a_j| B^j M_{n+j} \leqslant A' B^n M'_n,$$

where

$$A' = A\left(\sum_{j=0}^{d} |a_j|\, B^j\right), \qquad M'_n = \max_{0 \leqslant j \leqslant d} M_{n+j}.$$

Condition (ii) clearly holds for (M'_n), and to see (iv) we remark that a convex sequence, and a fortiori a log-convex sequence, is either decreasing or eventually increasing. In the first case we have $M'_n = M_n$, and in the second case we have $M'_n = M_{n+d}$ for $n \geqslant n_0$, so in both cases (iv) is clear.

The sequence $M''_n = M'_n/M'_0$ satisfies (i), (ii), and (iv) and $p(E)f \in \mathscr{F}\{M''_n\}$, i.e., $p(E)f$ is quasi-analytic. ∎

Remark. If f is exponentially bounded, then so is $p(E)f$ for any $p \in \mathbb{R}[X]$. The converse is also true but less obvious: If *$p(E)f$ is exponentially bounded for some $p \in \mathbb{R}[X]$, then f is exponentially bounded.* In fact, the power series $H(z) = \sum_0^\infty p(E)f(n)\, z^n$ has a positive radius of convergence. If $p(z) = \sum_{i=0}^d a_i z^i$, $a_d \neq 0$, then $z^d p(1/z) = \sum_{i=0}^d a_i z^{d-i}$ is different from zero in a neighbourhood of the origin so

$$G(z) = \frac{z^d H(z)}{z^d p(1/z)}$$

is holomorphic in a sufficiently small disc centered at the origin, hence

$$G(z) = \sum_0^\infty g(n)\, z^n,$$

where g is exponentially bounded. The equation

$$\left(\sum_{i=0}^d a_i z^{d-i}\right) G(z) = z^d H(z)$$

implies $p(E)g = p(E)f$ so that $f = g + h$, where h is of finite rank. In particular h and hence f is exponentially bounded.

PROPOSITION 3.5. *A quasi-analytic sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ with κ negative squares is polynomially determinate.*

Proof. By Lemma 3.4 it suffices to prove that a quasi-analytic positive definite sequence f is a determinate moment sequence. By assumption f satisfies

$$|f(n)| \leqslant AB^n M_n, \qquad n \geqslant 0,$$

where $(M_n)_{n \geqslant 0}$ verifies (i)–(iv). It is easy to see that $1/\sqrt[n]{M_n}$ is decreasing (cf. [18, p. 377]), so in addition to (iii) we have

$$\sum_0^\infty 1/\sqrt[2n]{M_{2n}} = \infty$$

and hence

$$\sum_0^\infty 1/\sqrt[2n]{f(2n)} = \infty.$$

By a theorem of Carleman it follows that $f(n)$ is a determinate moment sequence; cf. [20]. ∎

THEOREM 3.6. *Assume that* $f\colon \mathbb{N}_0 \to \mathbb{R}$ *has* κ *negative squares and is polynomially determinate.*

There is a unique minimal definitizing polynomial q *for* f, *and any definitizing polynomial* $\tilde{q}$ *for* f *has the form* $\tilde{q} = sq$, *where* s *is a non-negative monic polynomial.*

Proof. Let q_1, q_2 be two definitizing polynomials for f and assume q_1 minimal, hence of degree 2κ. Since q_1 and q_2 are both non-negative, it is clear that the greatest common monic divisor t of q_1 and q_2 is non-negative, and we have a factorization $q_i = ts_i$, where s_1, s_2 are monic and non-negative. By Theorem 1.3, $g := t(E)f$ has λ negative squares where $0 \leqslant \lambda \leqslant \kappa$, and $s_i(E)\,g$ is positive definite, $i = 1, 2$. Furthermore s_1, s_2 are relatively prime in $\mathbb{R}[X]$.

Since $s_i(E)\,g$ is positive definite, Hamburger's theorem yields the existence of a non-negative measure μ_i on $\mathbb{R}$ such that

$$s_i(E)\,g(n) = \int x^n \, d\mu_i(x), \qquad i = 1, 2.$$

We find

$$(s_1 s_2)(E)\,g(n) = \int x^n s_1(x)\, d\mu_2(x) = \int x^n s_2(x)\, d\mu_1(x),$$

but $(s_1 s_2)(E)\,g = (s_1 s_2 t)(E)\,f$ is a determinate moment sequence, because f is assumed polynomially determinate, hence

$$s_1 \mu_2 = s_2 \mu_1. \tag{3.2}$$

Since s_1, s_2 have no common real roots, we see that the open sets $G_i = \{x \in \mathbb{R} \mid s_i(x) > 0\}$, $i = 1, 2$, cover the whole real line. On G_i we define the Radon measure $\nu_i = (1/s_i)\,\mu_i$, $i = 1$, 2, and by (3.2) we have $\nu_1 = \nu_2$ on $G_1 \cap G_2$. By the principle of localization for Radon measures (cf., e.g., [3, p. 30]), there exists a unique Radon measure σ on $\mathbb{R}$ such that $\sigma \mid G_i = \nu_i$, $i = 1, 2$, and it is easily seen that $s_i \sigma = \mu_i$, and that σ has moments of every order. Defining

$$h(n) = \int x^n \, d\sigma(x)$$

we get

$$s_1(E)(g - h)(n) = \int x^n d(\mu_1 - s_1 \sigma) = 0$$

and similarly $s_2(E)(g - h) = 0$. This shows by Lemma 2.1 that $g \equiv h$, i.e., $g = t(E)\,f$ is positive definite, so t is a definitizing polynomial for f, hence of degree $\geqslant 2\kappa$. On the other hand, t is a divisor of q_1 and hence of degree $\leqslant 2\kappa$. We have now shown that the greatest common monic divisor t of q_1 and q_2 has the same degree as q_1 hence $q_1 = t$ and $q_2 = q_1 s_2$.

If also q_2 is minimal we get that $q_1 = q_2$. ∎

In sharp contrast to the uniqueness proved above we give the following result:

THEOREM 3.7. *Let q_1, q_2 be any two monic non-negative polynomials of degree 2κ. Then there exists a sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ with κ negative squares such that $q_1(E)f$ and $q_2(E)f$ are positive definite. In particular there exist distinct minimal definitizing polynomials for f.*

Proof. The proof uses well-known properties of Nevanlinna extremal solutions to an indeterminate moment problem; cf. [1, 20].

We first choose a compact set K containing the zeros of q_1 and q_2 and such that $2q_2 - q_1 > 0$ on $\mathbb{R}\backslash K$. Note that $2q_2 - q_1$ is monic of degree 2κ.

A Nevanlinna extremal measure is discrete with mass in countably many points, which are the zeros of a certain entire function. Given a compact set K and a Nevanlinna extremal measure μ we may therefore find an affine function φ such that the support of the image measure $\nu_1 = \varphi(\mu)$ is disjoint from K, but $\varphi(\mu)$ is again a Nevanlinna extremal measure. Next we choose another Nevanlinna extremal measure $\nu_2 \neq \nu_1$ with the same moments as ν_1 and whose support is also disjoint from K. This is possible because the Nevanlinna extremal measures associated with a fixed indeterminate moment sequence vary continuously with a real parameter.

We now define

$$\mu_j = (1/jq_j)\,\nu_j, \qquad j = 1, 2,$$

which are finite non-negative measures with moments of all orders since $q_j > 0$ on $\mathbb{R}\backslash K \supseteq \operatorname{supp}(\nu_j)$.

Finally, we put

$$f(n) = \int x^n\, d(\mu_1 - \mu_2), \qquad n \geqslant 0, \tag{3.3}$$

and claim that $q_j(E)f$ is positive definite, $j = 1, 2$.

Indeed

$$q_1(E)f(n) = \int x^n q_1(x)\, d(\mu_1 - \mu_2)(x) = \int x^n\, d\nu_1(x) - \int x^n \frac{q_1(x)}{2q_2(x)}\, d\nu_2(x)$$

$$= \int x^n \frac{2q_2(x) - q_1(x)}{2q_2(x)}\, d\nu_2(x),$$

where we have used the fact that ν_1 and ν_2 have the same moments. Since $2q_2 - q_1 > 0$ on $\mathbb{R}\backslash K \supseteq \operatorname{supp}(\nu_2)$ we get that $q_1(E)f$ is positive definite.

A similar calculation shows that

$$q_2(E)f(n) = \int x^n \frac{2q_2(x) - q_1(x)}{2q_1(x)} d\nu_1(x),$$

so that $q_2(E)f$ is also positive definite.

By Corollary 1.4 we see that f has λ negative squares where $0 \leqslant \lambda \leqslant \kappa$, and we will show that $\lambda = \kappa$. Assuming $\lambda < \kappa$ and letting q be a minimal definitizing polynomial for f of degree 2λ, then there exists a positive measure τ on $\mathbb{R}$ such that

$$q(E)f(n) = \int x^n \, d\tau(x).$$

On the other hand, by (3.3) we have

$$\begin{aligned} q(E)f(n) &= \int x^n q(x) \, d(\mu_1 - \mu_2)(x) \\ &= \int x^n \frac{q(x)}{q_1(x)} d\nu_1(x) - \int x^n \frac{q(x)}{2q_2(x)} d\nu_2(x). \end{aligned}$$

The rational function

$$d_j(x) = \frac{(1+x^2)\, q(x)}{jq_j(x)}$$

is bounded at infinity because the degree of the numerator is less than or equal to the degree of the denominator. Therefore d_j is bounded on the support of ν_j, which implies that the measure $\sigma_j = (q/jq_j)\, \nu_j$ has a bounded density with respect to $1/(1+x^2)\, \nu_j$. The latter measure is determinate, cf. [20, p. 64], and so is then σ_j. The above formulas for $q(E)f(n)$ show that σ_1 and $\tau + \sigma_2$ have the same moments, hence $\sigma_1 = \tau + \sigma_2$ or $\tau = \sigma_1 - \sigma_2$. But this contradicts τ being a positive measure since $\operatorname{supp}(\sigma_1) \cap \operatorname{supp}(\sigma_2) \subseteq \operatorname{supp}(\nu_1) \cap \operatorname{supp}(\nu_2) = \varnothing$.

This contradiction finally establishes that f has κ negative squares. ∎

The theory outlined so far shows that a sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ with κ negative squares is a solution to a non-homogeneous linear difference equation of order 2κ of the form

$$q(E)f(n) = \int x^n \, d\nu(x),$$

where q is a non-negative monic polynomial of degree 2κ and ν is a non-negative measure on $\mathbb{R}$ with moments of all orders.

Given a non-negative monic polynomial q of degree 2κ. The set of sequences $f\colon \mathbb{N}_0 \to \mathbb{R}$ for which $q(E)f$ is positive definite is a convex cone $\mathscr{C}_q$ of sequences with at most κ negative squares. We shall now give an integral representation of the elements in $\mathscr{C}_q$. This is a generalization of the Lévy–Khinchin formula for negative definite sequences; cf. [3, p. 188]. Our representation is contained in the integral representation by Krein and Langer; cf. [14, Theorem 2.1].

We first write $q = q_1 q_2$, where q_1 is the monic polynomial of highest degree having the same real roots as q. The degree of q_1 is even, say $2d_1$, and q_2 has no real roots. Let $\Gamma = \{t_1, ..., t_m\}$ be the real roots of q_1.

For $x \in \mathbb{R}$ let $l(x, n)$ be the sequence satisfying the difference equation

$$\begin{aligned} q_1(E)\, l(x, n) &= 0, \qquad n \geqslant 0, \\ l(x, n) &= x^n \qquad \text{for} \qquad n = 0, 1, ..., 2d_1 - 1. \end{aligned} \tag{3.4}$$

Since q_1 is monic we have $l(x, 2d_1) = x^{2d_1} - q_1(x)$, and it follows by recursion that $l(x, n)$ is a polynomial of degree $\leqslant 2d_1 - 1$ for all $n \geqslant 0$.

LEMMA 3.8. *For each $n \geqslant 0$ there exists $r(\cdot, n) \in \mathbb{R}[X]$ such that*

$$x^n - l(x, n) = q_1(x)\, r(x, n),$$

i.e., $l(x, n)$ is the remainder when x^n is divided by q_1.

Proof. We have that $g(x, n) = x^n - l(x, n)$ is a solution to the difference equation

$$\begin{aligned} q_1(E)\, g(x, n) &= q_1(x)\, x^n, \qquad n \geqslant 0, \\ g(x, n) &= 0, \qquad n = 0, 1, ..., 2d_1 - 1, \end{aligned}$$

and we see that $g(x, 2d_1) = q_1(x)$, so the formula holds for $n = 0, 1, ..., 2d_1$ with $r(x, n) = 0$ for $n \leqslant 2d_1 - 1$, $r(x, 2d_1) = 1$. The formula follows now easily by recursion and $r(x, n)$ is a polynomial of degree $n - 2d_1$ for $n \geqslant 2d_1$. ∎

THEOREM 3.9 (Generalized Lévy–Khinchin formula). *For each $f \in \mathscr{C}_q$ there exist:*

(i) *a non-negative Radon measure μ* on *$\mathbb{R} \setminus \Gamma$ satisfying*

$$\int_{\mathbb{R}\setminus\Gamma} q_1(x)\, x^{2n}\, d\mu(x) < \infty \qquad \textit{for} \quad n \geqslant 0,$$

(ii) *non-negative numbers* $a_1, ..., a_m$,

(iii) *a solution* $h: \mathbb{N}_0 \to \mathbb{R}$ *to the difference equation*

$$q(E)\,h(n) = \sum_{i=1}^{m} a_i t_i^n, \qquad n \geqslant 0,$$

such that

$$f(n) = \int_{\mathbb{R} \setminus \Gamma} (x^n - l(x, n))\, d\mu(x) + h(n), \qquad n \geqslant 0. \tag{3.5}$$

Conversely, for any $\mu, a_1, ..., a_m$, *and* h *as above the formula* (3.5) *determines a sequence* $f \in \mathscr{C}_q$.

Proof. For $f \in \mathscr{C}_q$ there exists a non-negative measure ν on $\mathbb{R}$ with moments of all orders such that

$$q(E) f(n) = \int x^n\, d\nu(x).$$

Let $\mu = (1/q)(\nu \,|\, (\mathbb{R} \setminus \Gamma))$, where $\nu \,|\, (\mathbb{R} \setminus \Gamma)$ denotes the restriction to $\mathbb{R} \setminus \Gamma$, and $a_i = \nu(\{t_i\})$, $i = 1, ..., m$. Then μ is a Radon measure satisfying (i), and by Lemma 3.8

$$f_1(n) = \int_{\mathbb{R} \setminus \Gamma} (x^n - l(x, n))\, d\mu(x), \qquad n \geqslant 0$$

is well defined and satisfies

$$\begin{aligned} q(E) f_1(n) &= \int_{\mathbb{R} \setminus \Gamma} (x^n q(x) - q_2(E)\, q_1(E)\, l(x, n))\, d\mu(x) \\ &= \int_{\mathbb{R} \setminus \Gamma} x^n\, d\nu(x) \end{aligned}$$

by (3.4) so

$$q(E)(f - f_1)(n) = \int_{\Gamma} x^n\, d\nu(x) = \sum_{i=1}^{m} a_i t_i^n,$$

showing that $h = f - f_1$ verifies (iii).

Conversely, if $\mu, a_1, ..., a_m$, and h are given satisfying (i)–(iii) we find

$$q(E) f(n) = \int x^n\, d\nu(x), \qquad n \geqslant 0$$

with $\nu = q\mu + \sum_{i=1}^{m} a_i \varepsilon_{t_i}$, which has moments of all orders because of (i). ∎

COROLLARY 3.10. *If q has no real roots, then the sequences $f \in \mathscr{C}_q$ are of the form*

$$f(n) = \int x^n \, d\mu(x) + h(n), \qquad n \geqslant 0,$$

where μ is a non-negative measure with moments of all orders and $h\colon \mathbb{N}_0 \to \mathbb{R}$ is a solution to the difference equation $q(E)\, h = 0$.

This is an immediate consequence of Theorem 3.9 since $q_1 = 1$, $\Gamma = \varnothing$.

For $q(x) = (x-1)^2$ we see that $\mathscr{C}_q$ consists of the conditionally positive definite sequences; cf. Section 1. In this case $q_1 = q$, $\Gamma = \{1\}$, $l(x, n) = 1 + n(x-1)$. The solutions of the difference equation

$$(E-1)^2\, h(n) = a$$

are given as

$$h(n) = \frac{1}{2} a n^2 + bn + c, \qquad b, c \in \mathbb{R}.$$

We have therefore given an alternative proof of the following Lévy–Khinchin representation proved by Horn in [7]. Another proof is given in [3, p. 188].

COROLLARY 3.11. *The conditionally positive definite sequences $f\colon \mathbb{N}_0 \to \mathbb{R}$ are given by the formula*

$$f(n) = \int_{\mathbb{R}\setminus\{1\}} (x^n - 1 - n(x-1))\, d\mu(x) + \frac{1}{2} a n^2 + bn + c,$$

where μ is a non-negative measure on $\mathbb{R}\setminus\{1\}$ such that

$$\int_{\mathbb{R}\setminus\{1\}} (x-1)^2\, x^{2n}\, d\mu(x) < \infty$$

and $a \geqslant 0$, $b, c \in \mathbb{R}$.

4. INTEGER VALUED SEQUENCES

Sequences $f\colon \mathbb{N}_0 \to \mathbb{Z}$ occur naturally in number theory, and if they are exponentially bounded, they may be studied in terms of their generating function

$$F(z) = \sum_0^{\infty} f(n)\, z^n. \tag{4.1}$$

A number of classical theorems assert that under suitable conditions F is a rational function or equivalently f is of finite rank; cf. Bieberbach [4, Sect. 6.2].

We are going to prove a result in the same direction but we formulate it as a *dichotomy result.*

THEOREM 4.1. *Assume that $f\colon \mathbb{N}_0 \to \mathbb{Z}$ satisfies*

$$|f(n)| \leqslant AB^n, \qquad n \geqslant 0,$$

where $A > 0$, $0 < B < 2$.

Then precisely one of the following two statements holds:

(i) $\operatorname{rk}(f) < \infty$.

(ii) $\kappa_-(f) = \kappa_+(f) = \infty$.

The following example due to T. Maack Bisgaard shows that in the above majorization $B < 2$ cannot be replaced by $B = 2$.

Let

$$f(n) = \frac{1}{2\pi} \int_0^{2\pi} (2\cos t)^n \, dt = \begin{cases} 0, & n \text{ odd}, \\ \binom{n}{n/2}, & n \text{ even}. \end{cases}$$

Then f is positive definite and satisfies

$$|f(n)| \leqslant 2^n, \qquad n \geqslant 0.$$

We next claim that f has infinite rank, for if not then

$$p(E) f(n) = \frac{1}{2\pi} \int_0^{2\pi} (2\cos t)^n p(2\cos t) \, dt = 0, \qquad n \geqslant 0$$

for any polynomial p which is divisible by the minimal polynomial for f, and this is clearly impossible if p is chosen $\geqslant 0$.

In the proof of Theorem 4.1 we need the following theorem of Polyà; cf. [4, p. 121]:

THEOREM. *Suppose $f\colon \mathbb{N}_0 \to \mathbb{Z}$ is such that the power series $\sum_0^\infty f(n) z^n$ has positive radius of convergence and that the sum $F(z)$ extends to a meromorphic function with only finitely many poles in a simply connected domain G with mapping radius (w.r. to zero) greater than one. Then F is a rational function. If in addition the radius of convergence of the power series is one then the poles of F are roots of unity.*

We also need the following operations $\#$ and $\square$ on sequences $f\colon \mathbb{N}_0 \to \mathbb{R}$. We define

$$f^{\#}(n) = \sum_{k=0}^{n} \binom{n}{k} 2^k f(n-k)$$

$$f^{\square}(n) = \sum_{k=0}^{n} \binom{n}{k} (-2)^k f(n-k), \qquad n \geqslant 0.$$

LEMMA 4.2. *For $f\colon \mathbb{N}_0 \to \mathbb{R}$ and a polynomial $p \in \mathbb{R}[X]$ we have*

(i) $(f^{\#})^{\square} = (f^{\square})^{\#} = f.$

(ii) $(p(E)f)^{\#} = p(E-2)f^{\#}$, $(p(E)f)^{\square} = p(E+2)f^{\square}$.

(iii) $\kappa_{\pm}(f) = \kappa_{\pm}(f^{\#}) = \kappa_{\pm}(f^{\square})$.

Proof. We use the fact that any sequence $f\colon \mathbb{N}_0 \to \mathbb{R}$ is the moment sequence of some signed measure σ, i.e.,

$$f(n) = \int x^n \, d\sigma(x), \qquad n \geqslant 0.$$

This is equivalent to an old theorem of Borel that $f(n) = \varphi^{(n)}(0)$ for a suitable C^∞-function with compact support; cf. [2, p. 168].

It follows that

$$f^{\#}(n) = \int (x+2)^n \, d\sigma(x) = \int x^n \, d\sigma * \varepsilon_2(x)$$

$$f^{\square}(n) = \int (x-2)^n \, d\sigma(x) = \int x^n d\sigma * \varepsilon_{-2}(x)$$

so (i) is clear. But (i) implies

$$(p(E)f)^{\#}(n) = \int (x+2)^n p(x) \, d\sigma(x) = \int x^n p(x-2) \, d\sigma * \varepsilon_2(x)$$

$$= p(E-2) f^{\#}(n)$$

and similarly $(p(E)f)^{\square}(n) = p(E+2) f^{\square}(n)$.

If f is positive definite, we may choose σ to be non-negative, and then $f^{\#}$ and $f^{\square}$ are again positive definite. If f has $\kappa_-(f) < \infty$ negative squares and q is a minimal definitizing polynomial for f, then $q(E-2)$ is a definitizing polynomial for $f^{\#}$ so $\kappa_-(f^{\#}) \leqslant \kappa_-(f)$ and similarly $\kappa_-(f^{\square}) \leqslant \kappa_-(f)$. Replacing f by $f^{\square}$ we get

$$\kappa_-(f) = \kappa_-((f^{\square})^{\#}) \leqslant \kappa_-(f^{\square}) \leqslant \kappa_-(f).$$

Finally, if $\kappa_-(f)=\infty$ then $\kappa_-(f^\#)=\kappa_-(f^\square)=\infty$ as a consequence of what has already been shown, and hence the proof of (iii) is completed by replacing f with $-f$. ∎

Proof of Theorem 4.1. It suffices to prove that $\operatorname{rk}(f)<\infty$ under the assumption that $\kappa=\kappa_-(f)<\infty$. Let q be a minimal definitizing polynomial for f and let σ be a non-negative measure on $\mathbb{R}$ such that

$$q(E)f(n)=\int x^n\,d\sigma(x).$$

We have an estimate of the form

$$|q(E)f(n)|\leqslant A_1B^n, \qquad n\geqslant 0,$$

and from this it easily follows that $\operatorname{supp}(\sigma)\subseteq[-B,B]$; cf. [3, p. 116]. By Lemma 4.2 we have

$$(q(E)f)^\#(n)=q(E-2)f^\#(n)=\int x^n\,d\sigma*\varepsilon_2(x)$$

or by setting $q_1(x)=q(x-2)$, $\sigma_1=\sigma*\varepsilon_2$,

$$q_1(E)f^\#(n)=\int x^n\,d\sigma_1(x) \tag{4.2}$$

and $\operatorname{supp}\sigma_1\subseteq[2-B,2+B]\subseteq\,]0,4[$.

The generating functions

$$P(z)=\sum_{n=0}^{\infty}(q_1(E)f^\#)(n)\,z^n$$

$$F^\#(z)=\sum_{n=0}^{\infty}f^\#(n)\,z^n$$

converge for $|z|<1/(2+B)$, and for these z we have

$$q_1(1/z)\,F^\#(z)=P(z)+r(1/z), \tag{4.3}$$

where $r\in\mathbb{R}[X]$ is of degree $\leqslant\deg q_1=2\kappa$ and without constant term. Clearly (4.2) implies

$$P(z)=\int_{2-B}^{2+B}\frac{1}{1-tz}\,d\sigma_1(t) \qquad \text{for } |z|<\frac{1}{2+B},$$

but the integral on the right-hand side shows that P has a holomorphic continuation to $\Omega=\mathbb{C}\setminus[(2+B)^{-1},(2-B)^{-1}]$. By (4.3) it follows that $F^\#$ is meromorphic in Ω with only finitely many poles.

Now consider (minus the Koebe function)

$$K(z)=\frac{-z}{(1-z)^2}$$

which maps the open unit disc D biholomorphically onto the slit plan $\mathbb{C}\setminus[\frac{1}{4}, \infty[$. Since $K(-1)=\frac{1}{4}$ there exists by continuity an $\varepsilon>0$ such that

$$K(D(-1, \varepsilon))\subseteq \Omega, \qquad \text{where} \quad D(-1, \varepsilon)=\{z\in\mathbb{C} \mid |z+1|<\varepsilon\}.$$

The composed function $F^{\#}\circ K$ is meromorphic in $G=D\cup D(-1, \varepsilon)$ with only finitely many poles. Since the power series of $F^{\#}$ and K have integer coefficients (and $K(0)=0$), the same holds for the power series of $F^{\#}\circ K$. The mapping radius of G w.r. to zero is greater than 1, so by Polyà's theorem $F^{\#}\circ K$ is a rational function. If z is replaced by $K(z)$ in (4.3), this formula immediately shows that $P\circ K$ is a rational function, so in particular $P\circ K$ has at most finitely many poles on the unit circle which by K is mapped onto $[\frac{1}{4}, \infty]$. This shows that P has at most finitely many poles on $[\frac{1}{4}, \infty[$, but the inversion formula for the Stieltjes transformation then shows that σ_1 is a discrete measure with finitely many atoms, so that P and hence $F^{\#}$ is a rational function. By Proposition 2.6 we get that $\operatorname{rk}(f)=\operatorname{rk}(f^{\#})<\infty$. ▮

As an example where Theorem 4.1 may be applied we take the sequence of primes $f(0)=1$, $f(1)=2$, $f(2)=3$, $f(3)=5$, ... By the prime number theorem $f(n)\sim n\log n$ so f satisfies the growth assumption with any $B>1$. That f is not of finite rank and hence $\kappa_-(f)=\kappa_-(f)=\infty$ follows from the next result.

THEOREM 4.3. *Let $f\colon \mathbb{N}_0\to\mathbb{Z}$ be an integer valued sequence and suppose that there exists a non-zero polynomial $p\in\mathbb{R}[X]$ such that $f(n)\sim p(n)\log n$ for $n\to\infty$.*

Then $\operatorname{rk}(f)=\kappa_+(f)=\kappa_-(f)=\infty$.

Proof. To derive a contradiction suppose that f is of finite rank. The rational function

$$F(z)=\sum_0^\infty f(n)\,z^n \tag{4.4}$$

can be written $F(z)=a(z)/b(z)$, where a, b are polynomials with integer coefficients and $b(0)=1$; cf. [19]. The growth hypothesis for f implies that the radius of convergence for (4.4) is one. By the earlier mentioned theorem of Polyà all roots of b are roots of unity, so the absolute value of their product is $|1/b_d| = 1$, where b_d is the leading coefficient of b and $d = \deg b$. It follows that $b_d = \pm 1$. The reciprocal polynomial to b is defined as

$$\tilde{b}(z) = z^d b(1/z) = \sum_{i=0}^{d} \tilde{b}_i z^i,$$

and we have

$$\tilde{b}(1/z)\, F(z) = a(z)\, z^{-d},$$

so

$$\sum_{i=0}^{d} \tilde{b}_i f(n+i) = 0 \qquad \text{for} \quad n > \deg(a) - d$$

and hence

$$\tilde{b}(E)\, E^{n_0} f = 0 \qquad \text{for} \quad n_0 = \deg(a) - d + 1.$$

The roots of b and $\tilde{b}$ are the same and lie on the unit circle. It follows from Section 2 that

$$E^{n_0} f(n) = \sum (p_i(n) \cos(n\theta_i) + q_i(n) \sin(n\theta_i)), \qquad n \geqslant 0,$$

where each term corresponds to a real root or a pair of complex conjugate roots of $\tilde{b}$.

Moreover p_i, q_i are real polynomials and θ_i is a rational multiple of 2π. If we choose $n_1 < n_2 < \cdots$ from $\mathbb{N}_0$ such that $n_k \theta_i \in 2\pi\mathbb{Z}$ for all k and i, we have

$$E^{n_0} f(n_k) = \sum p_i(n_k) \sim p(n_k + n_0) \log(n_k + n_0), \qquad k \to \infty$$

but this is clearly a contradiction. ∎

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